Combinatorial Designs Meet Hypercliques: Higher Lower Bounds for Klee's Measure Problem and Related Problems in Dimensions $d \geq 4$

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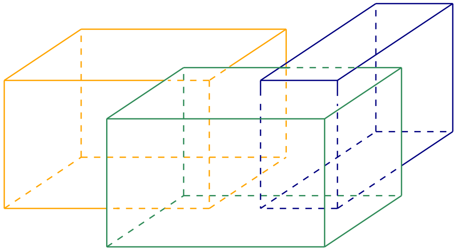
Klee's Measure Problem

Klee's Measure Problem (KMP)

Input: n axis-parallel boxes in $\mathbb{R}^d$.
Output: volume of the union of these boxes.



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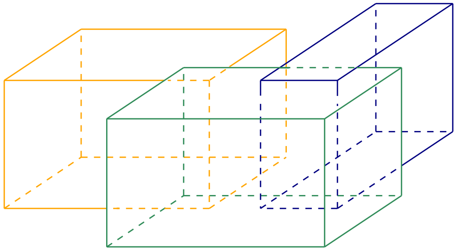
- basic geometric primitive
- many related problems, e.g.:
  - depth of axis-parallel boxes
  - largest empty (anchored) box
  - discrepancy of boxes
  - ...



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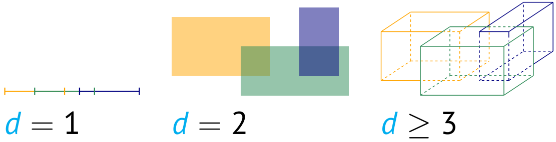

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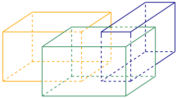

$d \geq 3$

[Klee '77] $O(n \log n)$

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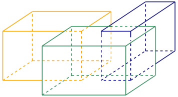

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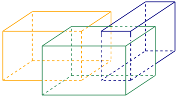

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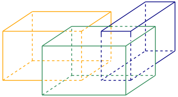

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[Overmars, Yap SICOMP'91]			$O(n^{\frac{d}{2}} \log n)$
[Chan Comp. Geom.'10]			$O(n^{\frac{d}{2}} 2^{O(\log^* n)})$
[Chan FOCS'13]			$O(n^{\frac{d}{2}})$

Klee's Measure Problem: Lower Bounds

- [Chan FOCS'13] $\rightarrow$ tight lower bound of $\Omega(n^{\frac{d}{2}-o(1)})$ for **combinatorial** algorithms under the k-clique hypothesis.
- Can we make these lower bounds hold for **general** algorithms?

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UB: [Chan FOCS'13]	$O(n^{1.5})$	$O(n^2)$	$O(n^{2.5})$	$O(n^{d/2})$
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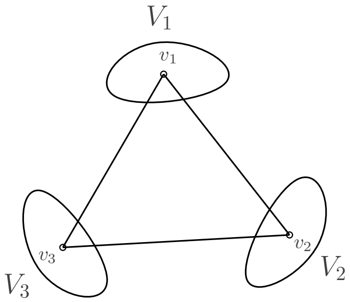
k-Clique Hypothesis

k-Clique

Input: k-partite graph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E)$, $|V_i| = n$ for all i.
Output: Does G have a clique of size k?
I.e., $v_1 \in V_1, \ldots, v_k \in V_k$, s.t. $\{v_a, v_b\} \in E$ for all $a \neq b$.

Best known algorithm $O(n^{\frac{\omega}{3}k})$ for k divisible by 3.



Combinatorial Clique Hypothesis

For any $k \geq 3$ there is no $O(n^{k-\varepsilon})$ combinatorial algorithm for k-Clique.

k-HyperClique Hypothesis

3-uniform k-HyperClique

Input: k-partite 3-uniform hypergraph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E)$, $|V_i| = n$ for all i.
Output: Does G have a hyperclique of size k?
I.e., $v_1 \in V_1, \ldots, v_k \in V_k$, s.t. $\{v_a, v_b, v_c\} \in E$ for all distinct a, b, c.

Best known algorithm $n^{k \pm o(1)}$ (essentially bruteforce).

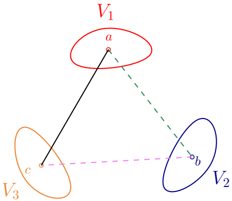
3-uniform HyperClique Hypothesis

For any $k > 3$ there is no $O(n^{k-\varepsilon})$ algorithm for 3-uniform k-hyperclique.

See [Lincoln, V. Williams, Williams'18], [Bringmann, Fischer, Künnemann'19], [Künnemann, Marx'20].

Chan's Combinatorial Lower Bound

Reduction from triangle detection.



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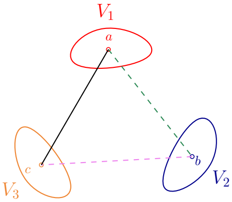

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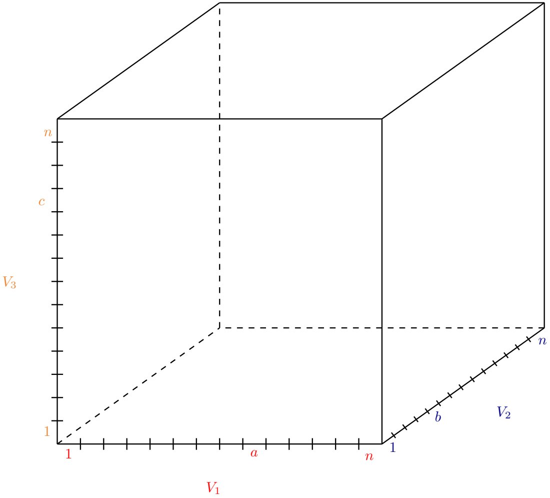

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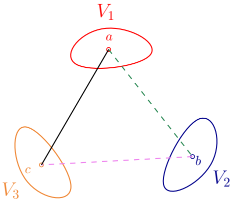
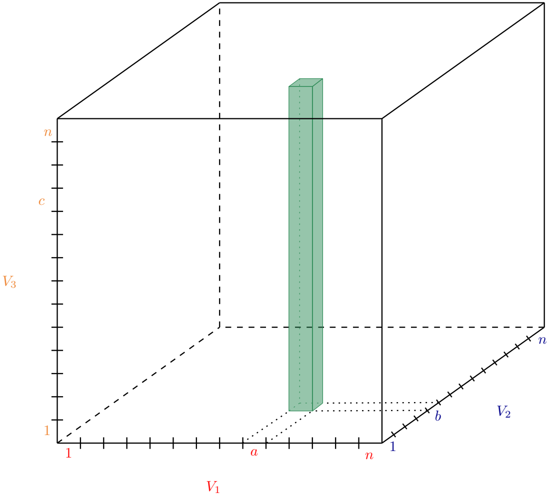


want: cube for (a, b, c) is covered by a box $\Leftrightarrow$ (a, b, c) do **not** form a triangle

- For all non-adjacent $a \in V_1$, $b \in V_2$ add a box covering all $(a, b, \cdot)$ unit cubes.



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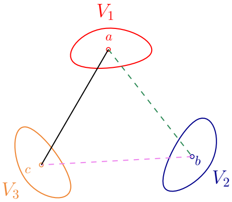

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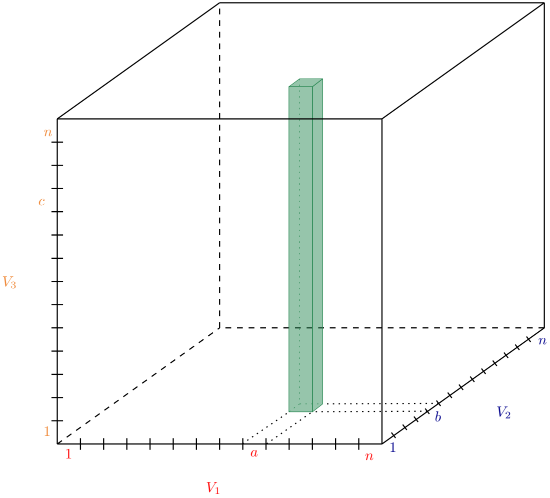
- For all non-adjacent $a \in V_1$, $b \in V_2$ add a box covering all $(a, b, \cdot)$ unit cubes.
- For all non-adjacent $a \in V_1$, $c \in V_3$ add a box covering all $(a, \cdot, c)$ unit cubes.



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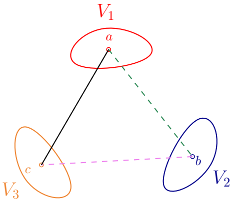

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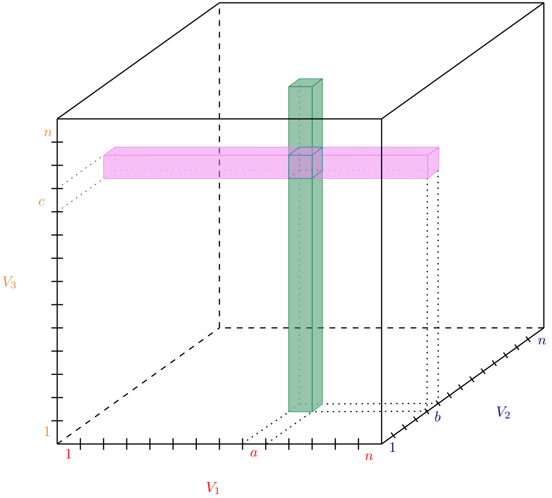

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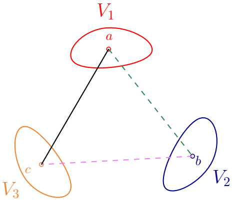

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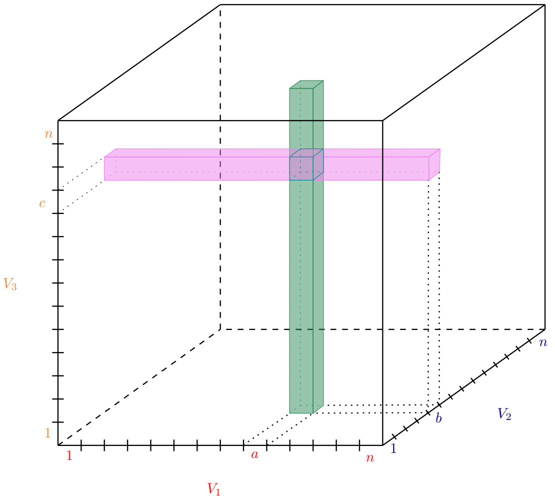
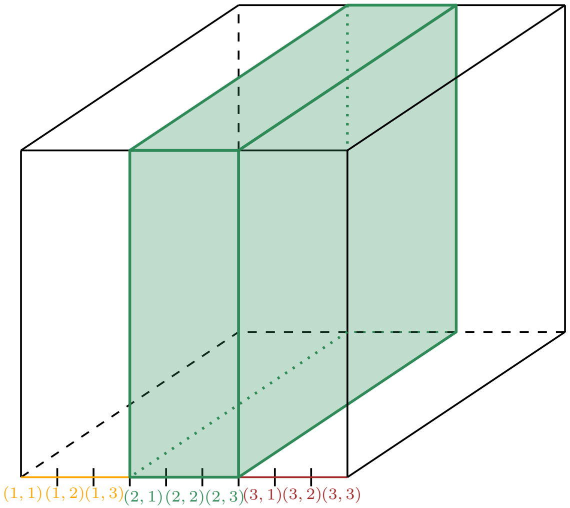
- We create $N = O(n^2)$ boxes.
- $\Omega(N^{\frac{d}{2}-o(1)})$ lower bound for combinatorial algorithms.



Lexicographic encoding

Encode a sequence of parts $(V_{i_1}, V_{i_2}, \ldots, V_{i_t})$ in each dimension lexicographically.

We can specify choices $(\cdot, \cdot, \ldots, v_{i_j}, \ldots, \cdot)$ as n^{j-1} boxes.



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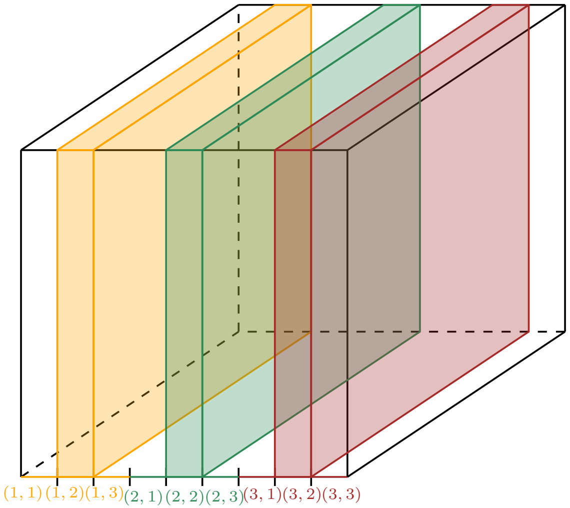
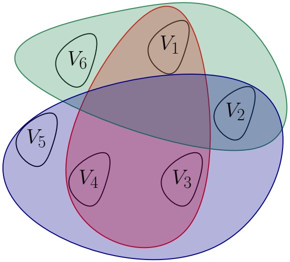
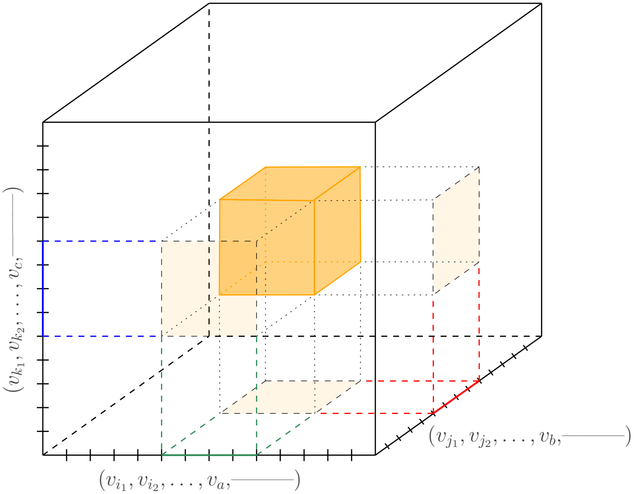


Redundant Encoding

Redundancy: encode parts multiple times in different dimensions.

1. $\forall v_a \in V_a, v_b \in V_b, v_c \in V_c$ s.t. $\{v_a, v_b, v_c\} \notin E$ add a box covering cubes corresponding to choosing this triplet into the hyperclique.





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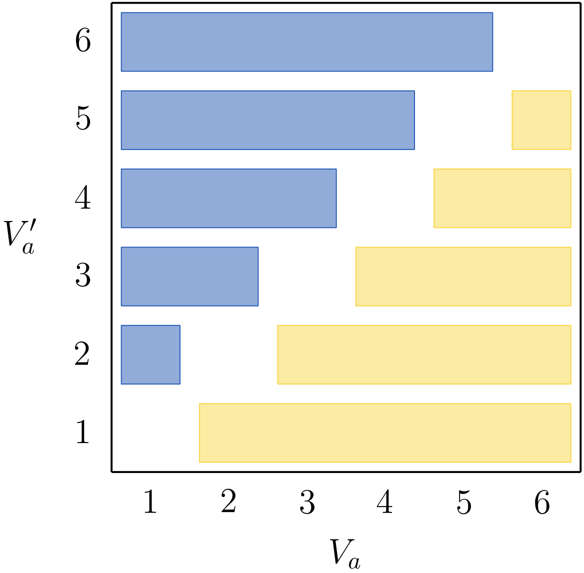
2. Cover inconsistent cubes: if for the same part different vertices are chosen in different dimensions.



Formalizing the setup

Definition: (d, k, α)-*prefix covering design* – d sequences over $\{1, \ldots, k\}$ s.t.:

$s_{1,1}\; s_{1,2}\; s_{1,3} \quad \cdots$
$s_{2,1}\; s_{2,2}\; s_{2,3} \quad \cdots$
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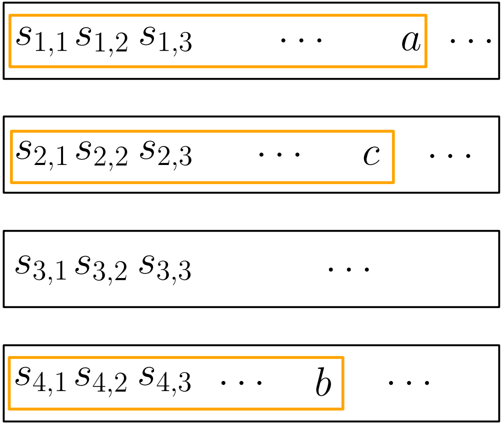

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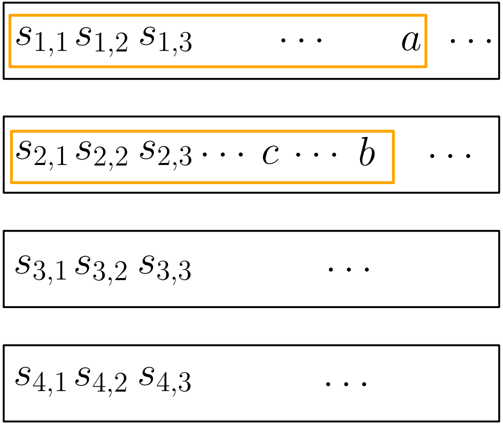

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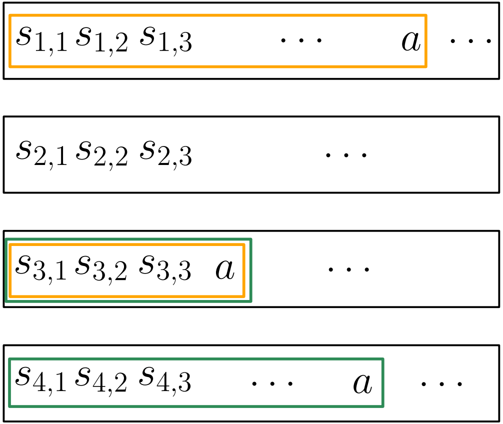

- **Triplet condition:** Every triplet of elements can be covered by 3 prefixes of total length $\leq \alpha$.
- **Singleton condition:** The first and last occurrences of any element can be covered by 2 prefixes of total length $\leq \alpha + 1$.

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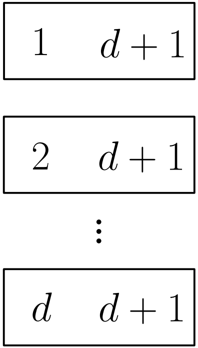

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$(d, d+1, 3)$:

$1 \quad d+1$

$2 \quad d+1$

$\vdots$

$d \quad d+1$

Theorem 1 (Helpful tool)

(d, k, α)-prefix covering design $\Rightarrow \Omega(N^{\frac{k}{\alpha} - o(1)})$ lower bound for KMP in $\mathbb{R}^d$ based on the 3-uniform k-hyperclique hypothesis.

Prefix Covering Designs: Examples

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- There exists a $(d, d^2, 3d - 3)$-prefix covering design giving an $\Omega(N^{\frac{d}{3}+\frac{1}{3}+\Theta(\frac{1}{d})})$ lower bound.

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Implicit in previous work [Künnemann FOCS'22]:

- $(3, 3g, 2g+1)$-prefix covering design for any $g \geq 1$:

1	2	$\cdots$	g	$2g$	$2g-1$	$\cdots$	$g+1$

$g+1$	$g+2$	$\cdots$	$2g$	$3g$	$3g-1$	$\cdots$	$2g+1$

$2g+1$	$2g+2$	$\cdots$	$3g$	g	$g-1$	$\cdots$	1

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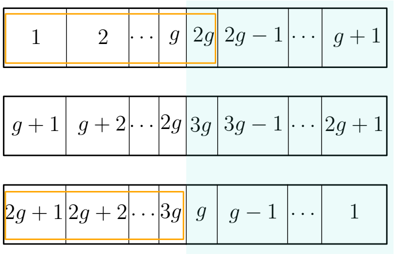

Prefix Covering Designs: Examples

Definition: (d, k, α)-*prefix covering design* – d sequences over $\{1, \ldots, k\}$ s.t.:

- Every triplet of elements can be covered by 3 prefixes of total length $\leq \alpha$.
- The first and last occurrences of any element can be covered by 2 prefixes of total length $\leq \alpha + 1$.

Implicit in previous work [Künnemann FOCS'22]:

- $(3, 3g, 2g + 1)$-prefix covering design for any $g \geq 1$:

1	2	$\cdots$	g	$2g$	$2g-1$	$\cdots$	$g+1$
$g+1$	$g+2$	$\cdots$	$2g$	$3g$	$3g-1$	$\cdots$	$2g+1$
$2g+1$	$2g+2$	$\cdots$	$3g$	g	$g-1$	$\cdots$	1

- $\Omega(N^{\frac{3g}{2g+1} - o(1)}) \xrightarrow{g \to \infty} \Omega(N^{\frac{3}{2} - o(1)})$ lower bound for $d = 3$.

Prefix Covering Designs: New Results for $d = 4, 5$

- Used the help of a SAT-solver to find good prefix covering designs.

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Theorem 2

There is a $(4, 40, 21)$-prefix covering design yielding an $\Omega(N^{1.9047\ldots - o(1)})$ lower bound for $\mathbb{R}^4$.

$$
\begin{aligned}
s_1 &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 40, 19, 28, 37, 26),\\
s_2 &= (11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 9, 38, 27, 36),\\
s_3 &= (21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 20, 39, 8, 7, 37),\\
s_4 &= (31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 10, 29, 18, 17, 27).
\end{aligned}
$$

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There is a $(5, 40, 18)$-prefix covering design yielding an $\Omega(N^{2.2222\ldots - o(1)})$ lower bound for $\mathbb{R}^5$.

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Theorem 4

Prefix covering designs cannot give a tight lower bound already for $d = 4$.

Prefix Covering Designs: New Results for all d

- **(v, c, 2) Covering Design** – collection of c-sized subsets $B_1, \ldots, B_d$ of the universe $\{1, 2, \ldots, v\}$ such that every couple of elements of the universe is fully contained in some B_i. (see La Jolla covering repository by D. M. Gordon: dmgordon.org/cover)

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Theorem 5 (Framework)

"Good" covering design with d subsets $\Rightarrow$ "good" prefix covering designs.
(+ a matching-like condition)

Prefix Covering Designs: New Results for all d

Theorem 6

Good specific lower bounds for fixed values of d + a general lower bound of $\Omega(N^{\frac{d}{3}+\frac{2}{9}\cdot\sqrt{d}-o(\sqrt{d})})$.

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Theorem 7

Prefix covering designs cannot give lower bounds higher than $N^{\frac{d}{3}+\sqrt{\frac{2}{9}}\cdot\sqrt{d}+o(\sqrt{d})}$.

Prefix Covering Designs: New Results for all d

Theorem 6

Good specific lower bounds for fixed values of d + a general lower bound of $\Omega(N^{\frac{d}{3}+\frac{2}{9}\cdot\sqrt{d}-o(\sqrt{d})})$.

Theorem 7

Prefix covering designs cannot give lower bounds higher than $N^{\frac{d}{3}+\sqrt{\frac{2}{9}}\cdot\sqrt{d}+o(\sqrt{d})}$.

[Künnemann FOCS'22] Lower Bound:	$\frac{d}{3}$	$+$	$\frac{1}{3}$		$+\Theta(\frac{1}{d})$
Our Lower Bound (Theorem 6):	$\frac{d}{3}$	$+$	$\frac{2}{9}$	$\cdot\sqrt{d}$	$-o(\sqrt{d})$
Limitation (Theorem 7):	$\frac{d}{3}$	$+$	$\sqrt{\frac{2}{9}}$	$\cdot\sqrt{d}$	$+o(\sqrt{d})$

Prefix Covering Designs: New Results Exponents Table

d	Upper bound [Chan'13]	Previous lower bound [Künnemann'22]	SAT-solver lower bound	Covering designs lower bound	(v, c) of the covering design
3	1.5	1.5		1.5	(3, 2)
4	2	1.777	1.9047	1.8461	(20, 12)
5	2.5	2.0833	2.2222	2.1929	(45, 25)
6	3	2.4		2.5714	(6, 3)
7	3.5	2.7222		3	(7, 3)
8	4	3.0476		3.3333	(24, 10)
9	4.5	3.375		3.6818	(90, 36)
⋮	⋮	⋮		⋮	⋮

Recap and Open Questions

Our Results:

- We showed lower bounds of $\Omega(N^{1.9047\ldots})$ in $\mathbb{R}^4$, $\Omega(N^{2.2222\ldots})$ in $\mathbb{R}^5$, and $\Omega(N^{\frac{d}{3}+\Theta(\sqrt{d})})$ in $\mathbb{R}^d$ for Klee's Measure Problem and related problems under the 3-uniform hyperclique hypothesis.
- These lower bounds are close to the best possible achievable using prefix covering designs.

Open Questions:

- Can we solve Klee's Measure Problem faster for large d?
- Can we show tight bounds for $d = 4, 5, 6$ using some different method?
- Can we show lower bounds of form $\Omega(N^{\gamma \cdot d - o(d)})$ for $\gamma > 1/3$?